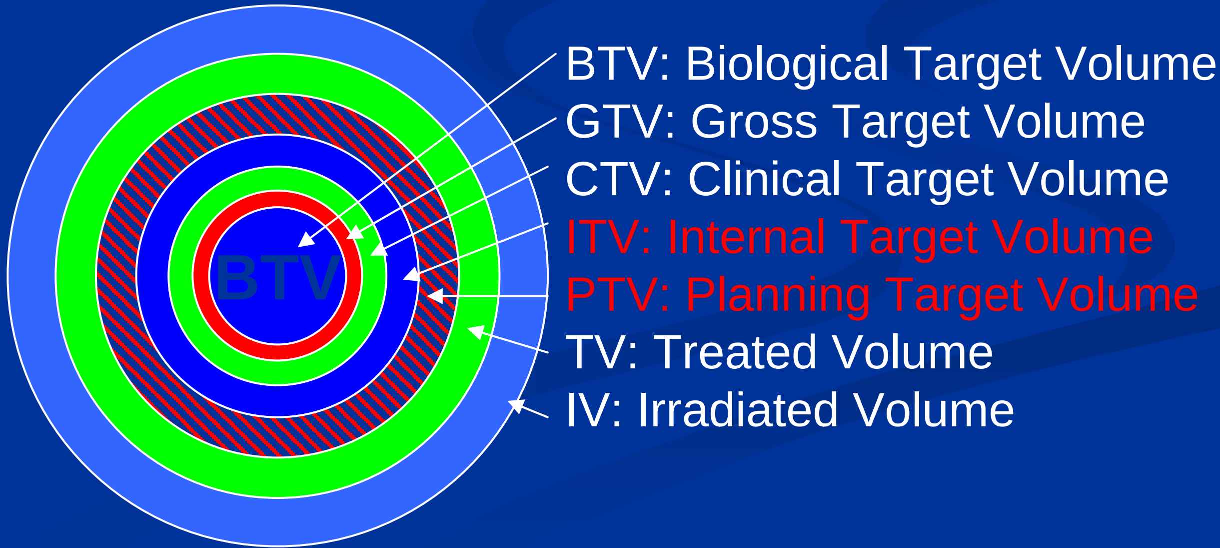
Modern methods for solving inverse problems

Jürgen Hesser
Experimental Radiation Oncology
University Medical Center Mannheim
University of Heidelberg

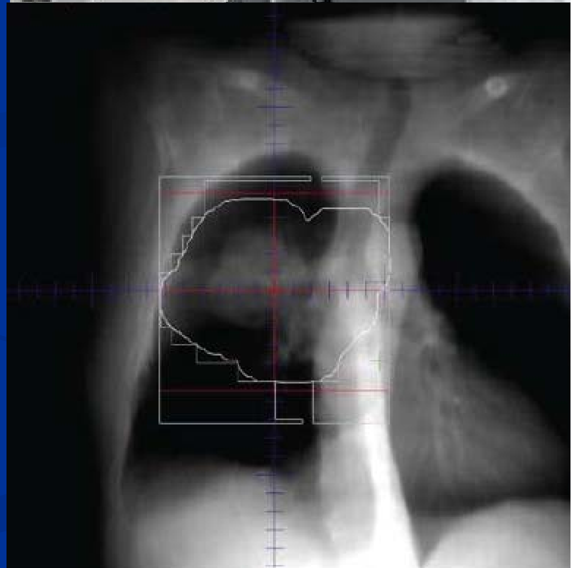
Motivation

- Image Guided Radiation Therapy (IGRT) allows to reduce the safety margins of treatment
  - less complications, better tumor control
  - new: on-board cone-beam CT



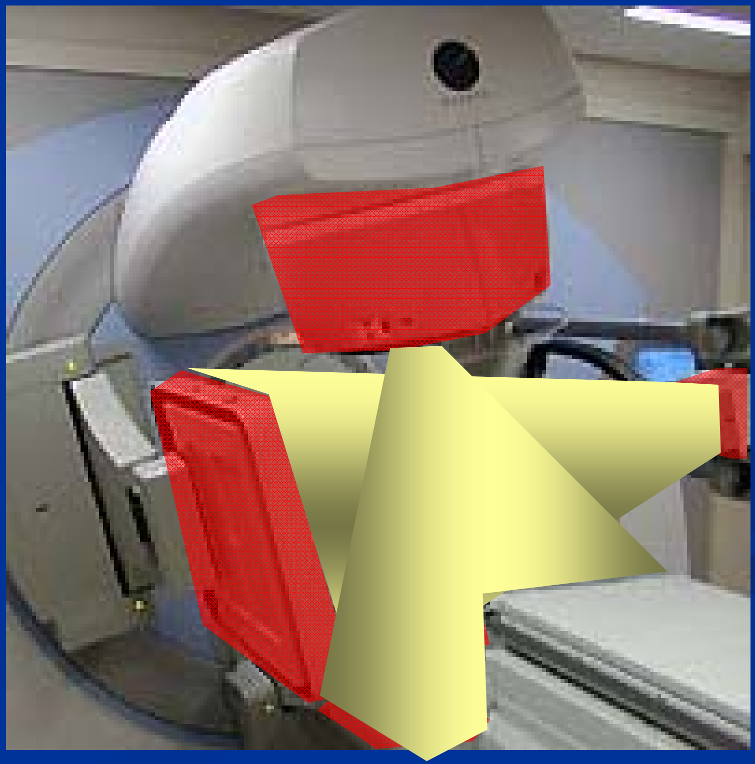
Motivation

- Acquisition requires 2 minutes scan time
- Lung cancer patients are treated in breath-hold:
  - requires acqisition times < 20s



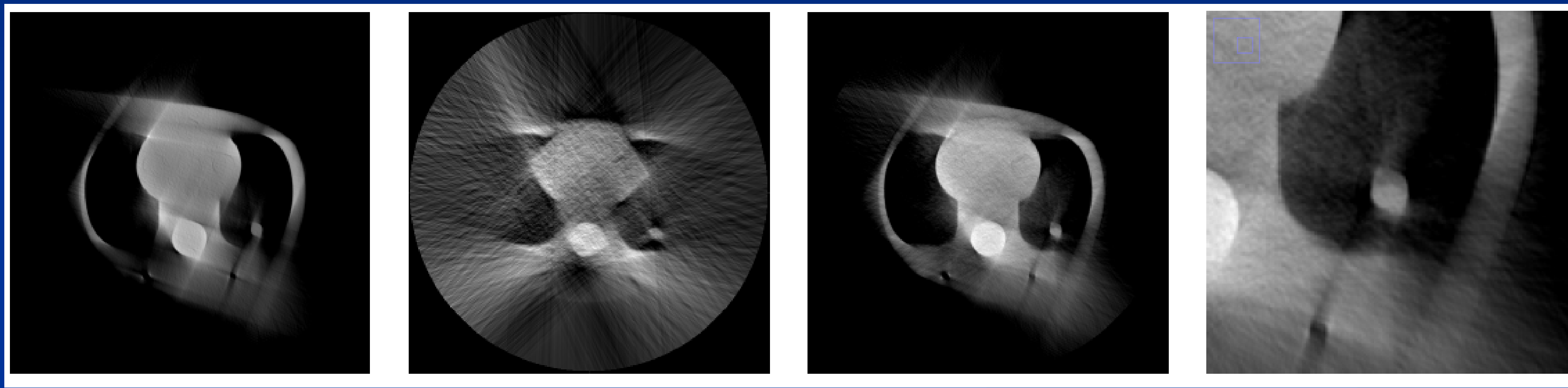
kV+MV-Reconstruction

- Solution: use both irradiation sources + faster rotation and < 180° angle
  - 15 s acquisition time
- Acquire kV+MV images
- Convert MV to kV
- Reconstruct



http://www.elekta.com/healthcare_international_beaumont_work_results_breakthrough.php

Recent Results
(Filtered Backprojection)



Question: How can such reconstruction be realized optimally?

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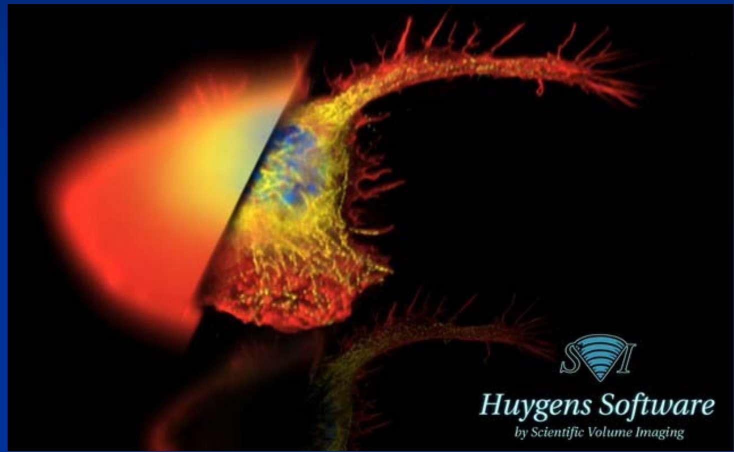
Examples of Inverse Problems

- Deblurring

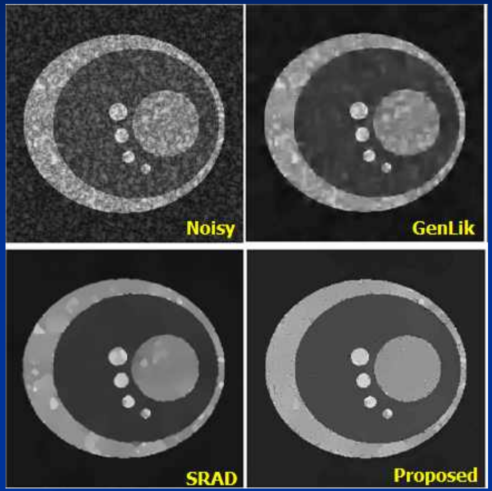


- Tomography

Examples of Inverse Problems

- Denoising

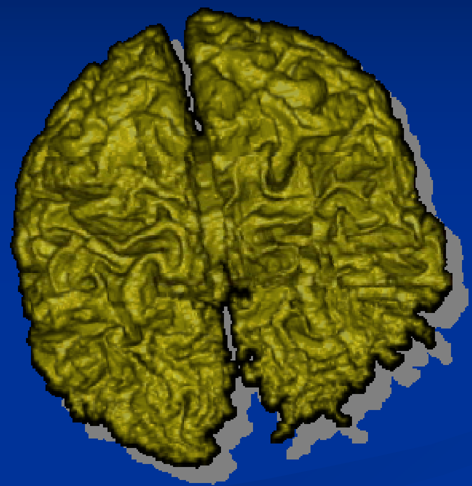
Y. Yue, M.M. Croitoru, A. Bidani, J.B. Zwischenberger, and J.W. Clark Jr.,Nonlinear Multiscale Wavelet Diffusion for Speckle Suppression and Edge Enhancement in Ultrasound Images, *IEEE Trans. Med. Imaging.* 2006 Mar;25(3):297-311



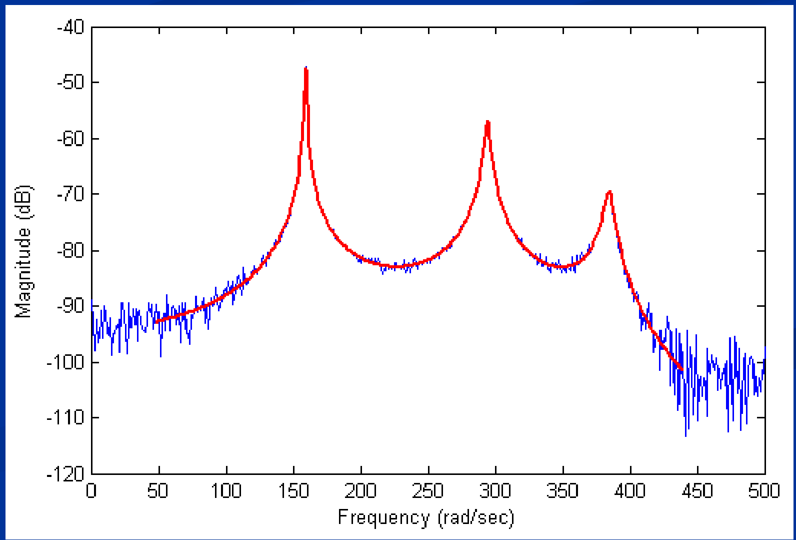
- Image registration

Examples of Inverse Problems

- Image Segmentation



- Parameter Estimation



...seems that most problems are inverse?

- ...but some of these are really difficult to solve
  - ill-posed inverse problems

Definition of Ill-Posed Inverse Problems

- Inverse problem
  - given: measurement g, functional W, free parameter/original data : f
    - W(f)=g

- A problem is called ill-posed if <u>at least one</u> of the following conditions is not fulfilled
  1. $\exists$f: W(f)=g for each g $\in$Y
  2. the solution f is unique
  3. the solution f depends on the data like a continuous function

Examples of Ill-posed Problems

- computer tomography
  - X-ray projection from a set of directions
  - compute volume
- model-fitting
- denoising
- deconvolution
  - remove blurring
- gridding and regridding
  - measure on discrete points, how to interpolate in between
- matching/registration
- segmentation
- numerical analysis: e.g. Fredholm equation

many problems in image analysis are ill-posed inverse problems

$$\int_a^b k(x,s) f(s) ds = d(x); a \le x \le b$$

...seems that most interesting inverse problems are ill-posed

- what to do?
- ...take the example of a cone-beam tomography

X-Ray Cone-Beam Tomography

http://www.medical.toshiba.com/images/products/cv/cv_DP-i_lg.jpg

X-Ray Cone-Beam Tomography

- Lambert-Beer absorption law (homogenous tissue)

$$I(x) = I(0)e^{-f \cdot x}$$

- absorption law for inhomogeneous tissue along a ray

$$I(x) = I(0)e^{-\int_0^x f(x')dx'}$$

$$P_j(x) = -\ln\frac{I(x)}{I(0)} = \int_0^x f(x')dx' = \int_0^x f(ray_j(x'))dx'$$

$$= \int_{\Omega:\text{object}} f(x,y)w_j(x,y)dxdy = \sum_{i,k=0}^{N-1} f(x_i,y_k)w_j(x_i,y_k)$$

$$= \sum_{i,k=0}^{N-1} f_{i,k}w_{j;i,k} = \left(W\vec{f}\right)_j$$

X-Ray Tomography

$$\vec{P} = W\vec{f}$$

solve

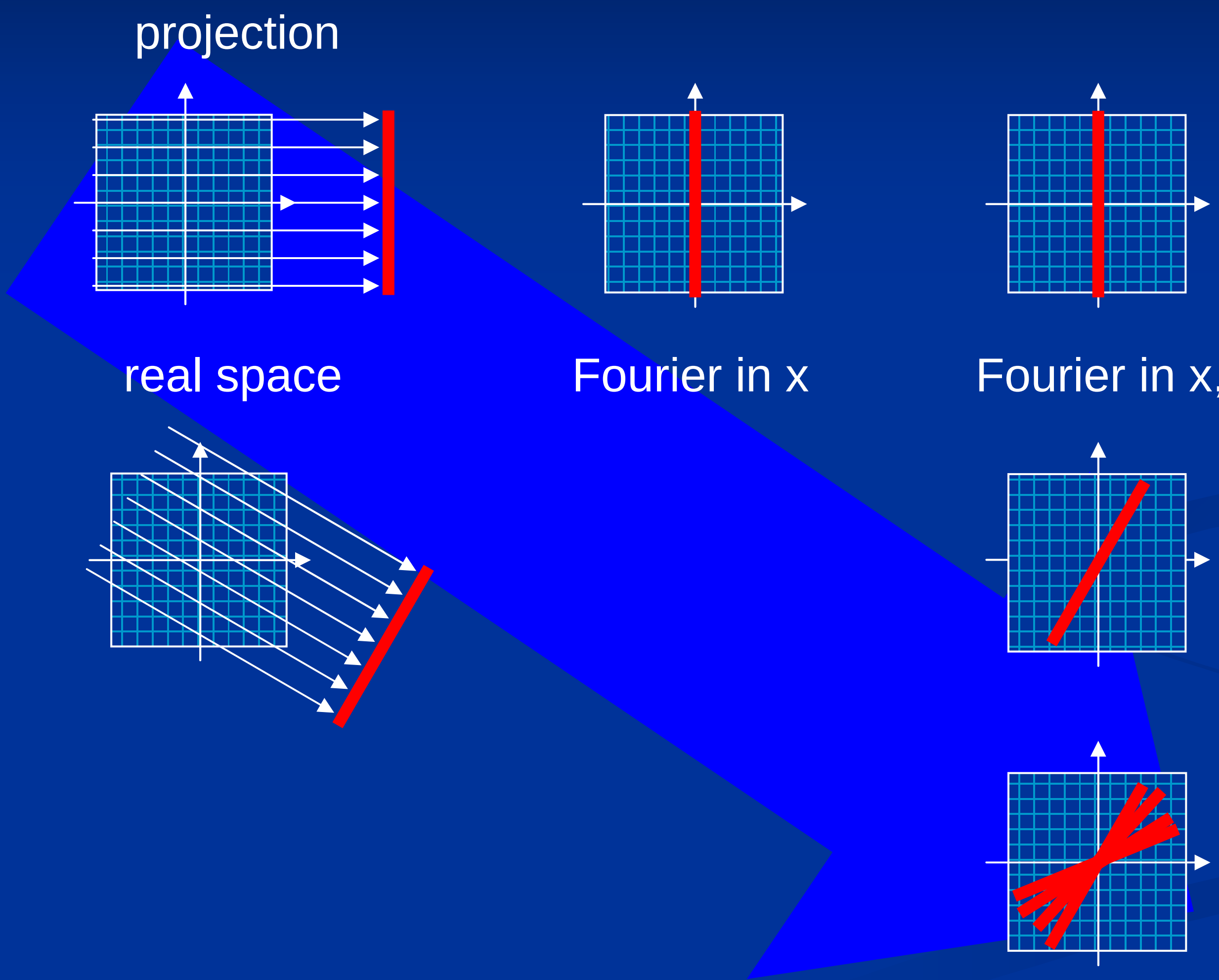
$$\vec{f} = W^{PI}\vec{P} = \left(W^{T}W\right)^{-1} W^{T}\vec{P}$$

resp.

$$M(\vec{f}) = \left\|\vec{P} - W\vec{f}\right\|^{2} \to \min$$

$$\vec{P} \in \Re^{n}; W \in \Re^{n\times n}$$

Filtered Backprojection



- generated projections from all directions
  - fills Fourier space densely
- projections given in cylinder coordinates (r,θ)

Filtered Backprojection

- Accumulate all projections in Fourier Space
  - regrid in Eucledian Space
  - apply inverse FFT
- Alternative
  - Filter in Real Space
  - Backproject
  - Accumulate projections

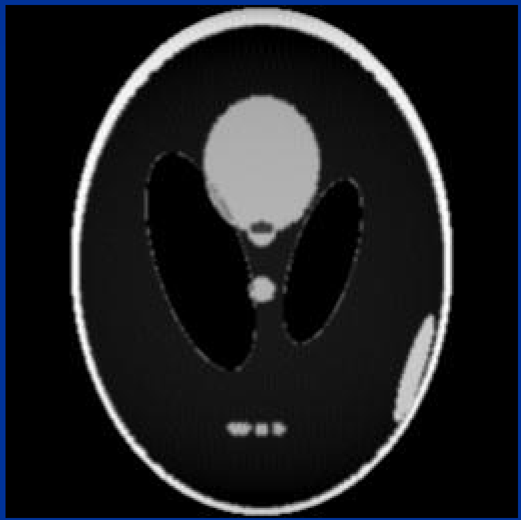
Example Filtered Backprojection

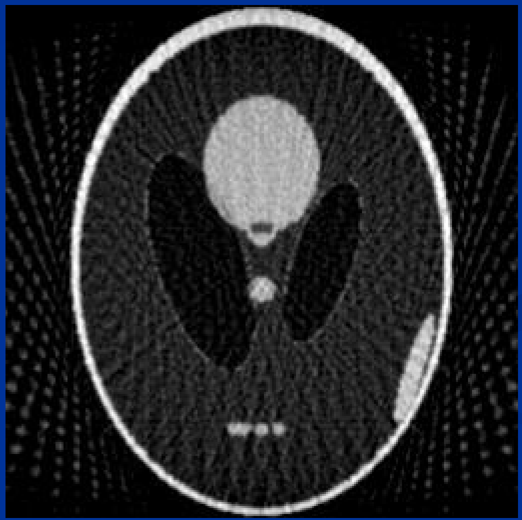
Image Size: 256 ×256

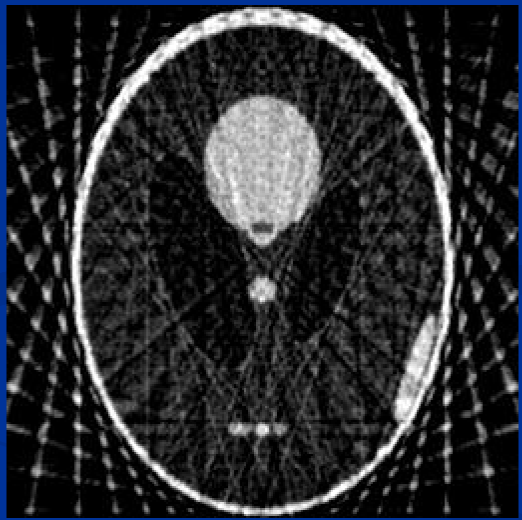
- many projections

few

very few







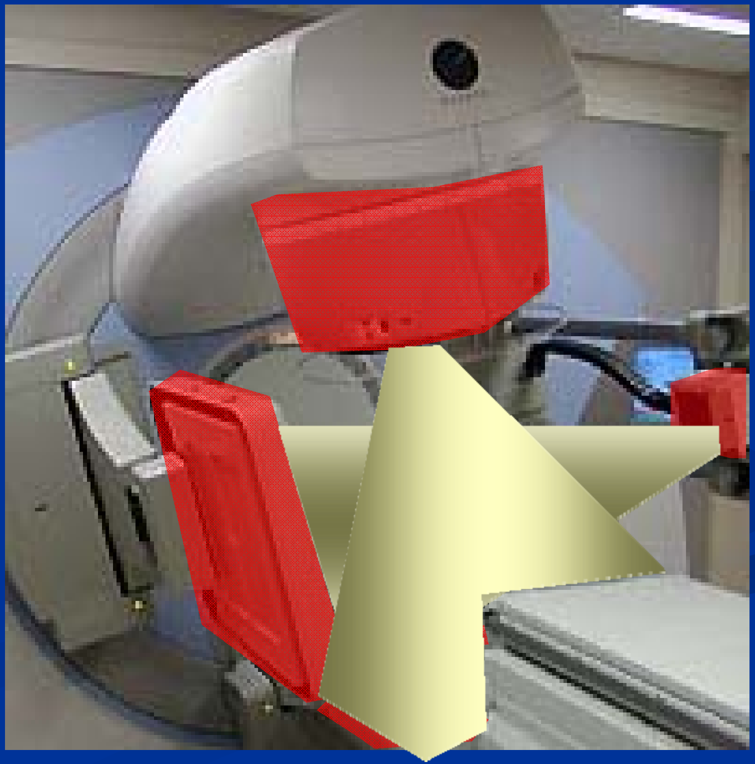
360

72

36

Problem

- Many projections required to fill frequency space
- Improvement:
  - Iterative Reconstruction



http://www.elekta.com/healthcare_international_beaumont_work_results_breakthrough.php

Iterative Reconstruction

- Idea:
  - numerically solve

  (e.g. gradient or conjugate gradient technique)

$$M(\vec{f}) = \left\| \vec{P} - W\vec{f} \right\|^2 \rightarrow \min$$

- Theory:
  - ...says that compared to Filtered Backprojection, only half of the number of projections is really required
- Cost
  - approx. 10x Filtered Backprojection

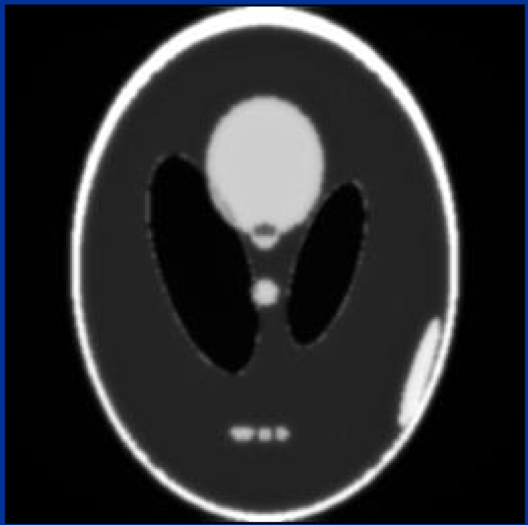
Example: SIRT

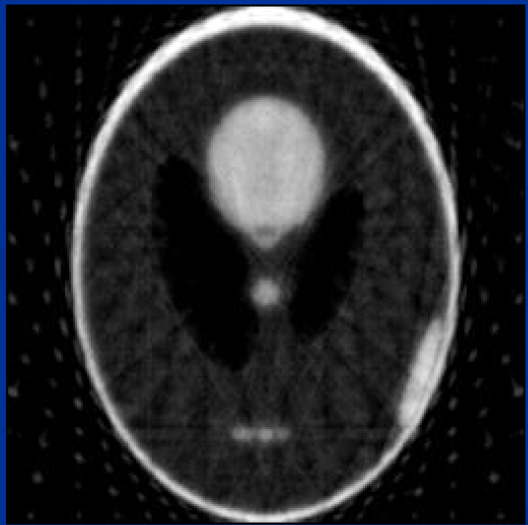
Image Size: 256 $\times$256

- many projections

few

very few





360
35 iterations

72
100 iterations

36
100 iterations

Stochastic Reconstruction

- Before:
  - stochastic noise not considered (Poisson noise)
    - Stochastic Reconstruction

Stochastic Reconstruction

- Idea: Bayes-Theorem

$$p(f \mid P) = \frac{p(P \mid f)\, p(f)}{p(P)}$$

likelihood $p(P \mid f)$ forward projection
p(): Poisson Distribution

- Approach: maximize

$$p(P \mid f) \quad p(f),\, p(P) \text{ assumed constant}$$

- Problem: typically does not consider *p*(Volume)

Stochastic Reconstruction

- Popular method
  - Expectation Maximization
    - E Step: calculate expectation, i.e. likelihood(f) given measurements, knowing previous volume;

$$E(f^{(k)}) = \left\langle L(v) \mid P_o, f^{(k)} \right\rangle$$

    - M Step: maximize likelihood

$$\frac{\partial E(f^{(k)})}{\partial f} = 0$$

- Problem: compute intensive
  - OSEM: ordered subset expectation maximization
    - processes a selection of projections at a time – accelerates processing

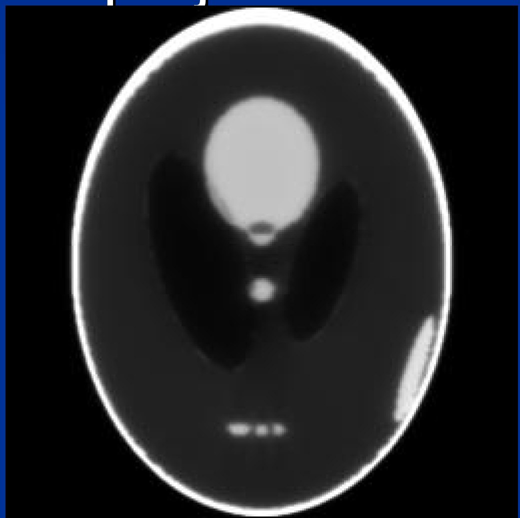
Example: OSEM

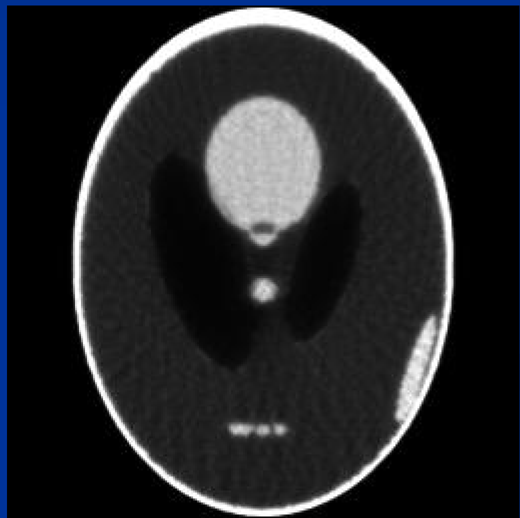
Image Size: 256 $\times$256

- many projections

few

very few





360	72	36
10 iterations	15 iterations	30 iterations
3 subsets	3 subsets	3 subsets

Regularized Stochastic Reconstruction

- Approach: Maximum A Posteriori

$$L(f) = \ln p(f \mid P) = \ln p(P \mid f) + \ln p(f) - \ln p(P)$$

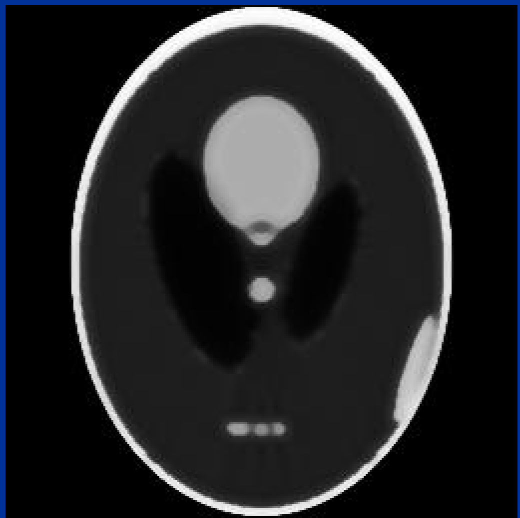
a posteriori likelihood a priori

- A Priory knowledge
  - smooth result
    - „linear theory“: Tikhonov Regularization $T(f) = \lambda \left|\nabla f\right|^2$
    - „non-linear theory“: non-linear regularizers like total variation $TV(f) = \lambda \left|\nabla f\right|$

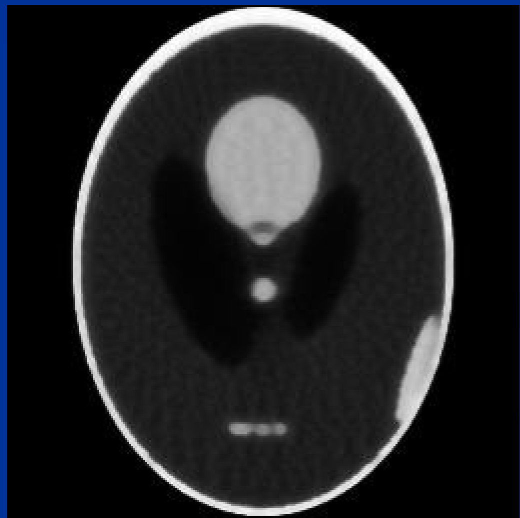
Example: OSEM with TV

Image Size: 256×256

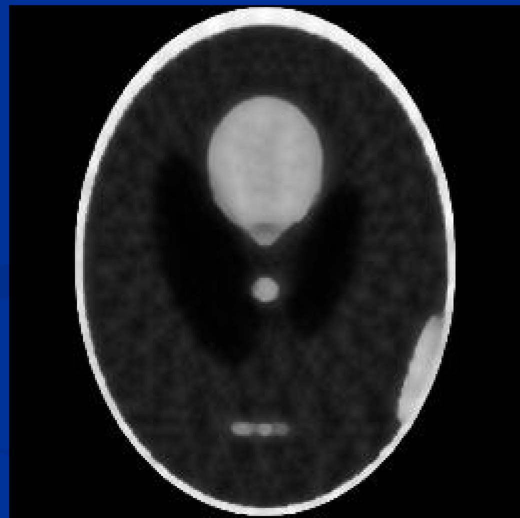
- many projections | few | very few



360
15 iterations



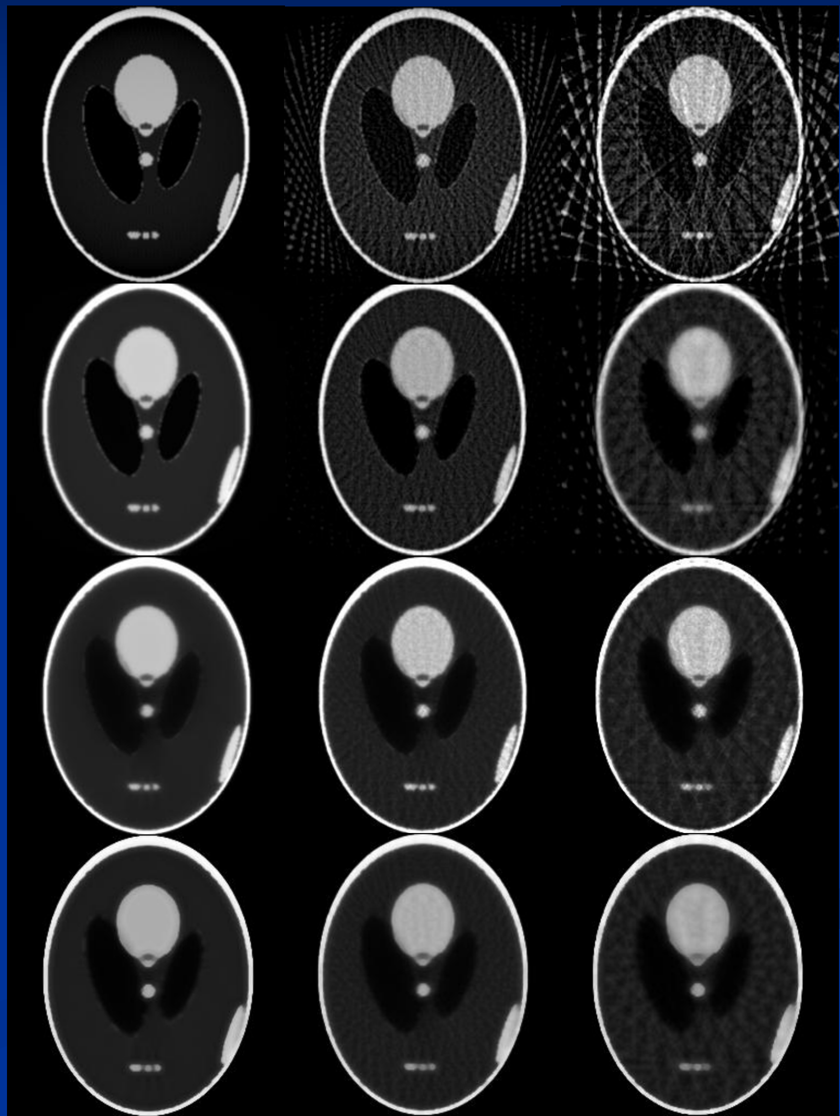
72
15 iterations



36
30 iterations

Comparison

- Filtered Backprojection
- Iterative Reconstruction
- Stochastic Reconstruction
- Regularized Stochastic Reconstruction



360 72 36

GPUs

- Use highly parallel systems for acceleration
  - GPUs = many simple CPUs
    - concentrate only on compute elements
    - leave out all control (superscalar hardware, look-up tables for caching, flow control)
    - consequence: programmer is restricted
      - many independent processes: single instruction multiple data

GPU-Acceleration

- Tomography reconstruction is time consuming for typical CT data sets
  - distribute compute tasks to compute nodes
    - independent computation
    - highly parallel

Mapping Tomography on GPUs

- Forward Projection
  - Graphics processing: sample along ray and accumulate
- Backprojection
  - Graphics processing: distribute ray intensity on voxels
- Regularization
  - For each voxel compute regularization gradient
- Update each voxel independently

GPU-Performance

- Forward projection (360 projections):
  - Volume 256^3, projection 256x256 = 4s
  - Volume 512^3, projection 512x512 = 28s
- Backward projection (360 projections):
  - Volume 256^3, projection 256x256 = 3s;
  - Volume 512^3, projection 512x512 = 12s;
- Iterative (GPU+CPU) (36 projections, 100 iterations):
  - non-regularized = 6 min;
  - Regularized = 30 min;

Outlook

- Solving inverse problems is a highly complex task
  - direct solution methods like filtered backprojection have severe limitations
  - iterative approaches improve considerably, especially when using stochastic approaches and regularization
- GPU implementation can mitigate the high compute intensivity